Consequently, in view of the theorem on an implicit function /5/, for small ε in the space A there is a solution $S(r, \varphi, \varepsilon)$ of the equation $F(S(r, \varphi, \varepsilon), \varepsilon) = 0$, which differs only slightly from S_0. The theorem is proved.

Returning to the old variables x_1 and x_2, we obtain at least a function of the class C^m. The solutions obtained, as in Sect.2, define the manifolds $y = \pm\partial S/\partial x$ of the phase trajectories of asymptotic motions (compare with /2/).

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Translated by R.C.G.

PMM U.S.S.R.,Vol.51,No.2,pp.182-189,1987
Printed in Great Britain
0021-8928/87 $10.00+0.00

PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS IN CERTAIN DEGENERATE CASES*

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Periodic solutions of a canonical system of differential equations with a special type Hamiltonian, i.e. of the so-called fundamental problem of dynamics /1/, are investigated. A method of constructing the conditions of periodicity of the solutions is given and a non-linear analysis of the solutions is carried out. The method enables the Poincare's classical conditions of existence, as well as of the new conditions of existence of periodic solutions in degenerate cases to be derived. The cases of degeneracy discussed here appear very frequently in various problems of dynamics. The results obtained are illustrated by finding new periodic solutions for the problem of the motion of a heavy rigid body about a fixed point.

1. Formulation of the problem. Consider the following system of canonical differential equations:

$$\frac{d\mathbf{I}}{dt} = \frac{\partial H}{\partial \boldsymbol{\varphi}^T}, \quad \frac{d\mathbf{J}}{dt} = \frac{\partial H}{\partial \boldsymbol{\psi}^T}, \quad \frac{d\boldsymbol{\varphi}}{dt} = -\frac{\partial H}{\partial \mathbf{I}^T}, \quad \frac{d\boldsymbol{\psi}}{dt} = -\frac{\partial H}{\partial \mathbf{J}^T} \tag{1.1}$$

$$\mathbf{I} = (p_1, \ldots, p_l)^T, \quad \mathbf{J} = (p_{l+1}, \ldots, p_N)^T, \quad \mathbf{p} = (p_1, \ldots, p_N)^T = (\mathbf{I}, \mathbf{J})^T$$

$$\boldsymbol{\varphi} = (q_1, \ldots, q_l)^T, \quad \boldsymbol{\psi} = (q_{l+1}, \ldots, q_N)^T,$$

$$\mathbf{q} = (q_1, \ldots, q_N)^T = (\boldsymbol{\varphi}, \boldsymbol{\psi})^T$$

$$H(\mathbf{p}, \mathbf{q}, t, \mu) \equiv H_0(\mathbf{I}) + \mu H_1(\mathbf{p}, \mathbf{q}, t) + \ldots, \quad |\mu| \ll 1 \tag{1.2}$$

Let H be an analytic function of the position variables $\mathbf{p}$, the canonical angle variables $\mathbf{q}$ and the time t, in the region $D \times T^N \times T^1$, where D is a bounded connected region $R^N\{p_1, \ldots, p_N\}$ of the N-dimensional plane, $T^N\{q_1, \ldots, q_N \bmod 2\pi\}$ is an N-dimensional torus and $T^1\{t \bmod T_0\}$. Then the functions $H_i(\mathbf{p}, \mathbf{q}, t)$ $(i \geqslant 1)$ can be expanded in convergent Fourier series over the multiple angle variables q and Ωt ($\Omega = 2\pi/T_0$ is the fundamental frequency and T_0 is the period)

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*Prikl.Matem.Mekhan.,51,2,235-243,1987

$$H_i(\mathbf{p},\mathbf{q},t)=\sum_{\|\mathbf{k}\|+|k_{N+1}|\geqslant 0} H_i^{(\mathbf{k},\,k_{N+1})}(\mathbf{I},\mathbf{J})\cos(\mathbf{k}^{(1)}\boldsymbol{\varphi}+\mathbf{k}^{(2)}\boldsymbol{\psi}+k_{N+1}\Omega t)+ \tag{1.3}$$
$$H_i^{*(\mathbf{k},\,k_{N+1})}(\mathbf{I},\mathbf{J})\sin(\mathbf{k}^{(1)}\boldsymbol{\varphi}+\mathbf{k}^{(2)}\boldsymbol{\psi}+k_{N+1}\Omega t)$$
$$\mathbf{k}^{(1)}=(k_1,\ldots,k_l),\quad \mathbf{k}^{(2)}=(k_{l+1},\ldots,k_N),\quad \mathbf{k}=(\mathbf{k}^{(1)},\mathbf{k}^{(2)})$$
$$k_i\in Z\quad (i=1,\ldots,N+1),\quad \|\mathbf{k}\|=\|\mathbf{k}^{(1)}\|+\|\mathbf{k}^{(2)}\|=\sum_{i=1}^{N}|k_i|$$

Eqs.(1.1)-(1.3) represent an important example of a system of differential equations close to integrable which frequently arise in celestial mechanics.

The right-hand sides of Eqs.(1.1)-(1.3) are T_0-periodic functions of time. The fundamental problem studied in this paper is whether the equations have periodic solutions with period T which is a multiple of T_0, at least for sufficiently small values of the parameter μ.

Poincaré obtained the sufficient condition for the existence of periodic solutions of Eqs.(1.1)-(1.3) in a form suitable for practical applications. However, in a number of practical cases (e.g. in problems of the translational and rotational motion of satellites) the classical Poincaré conditions do not hold. The question of the existence of periodic solutions in such cases (in what follows we shall call these cases singular or degenerate) remains, generally speaking, an open one.

Below we study the cases when the Poincaré-averaged function H_1 depends on the generating values of some of the variables only. Similar cases are encountered in many problems of celestial mechanics, for example in the study of high-order resonances.

When $\mu=0$, (1.1)-(1.3) yields a generating (unperturbed) system of equations whose general solution is

$$\mathbf{I}=\mathbf{I}_0,\quad \mathbf{J}=\mathbf{J}_0,\quad \boldsymbol{\varphi}=\mathbf{n}t+\boldsymbol{\varphi}_0,\quad \boldsymbol{\psi}=\boldsymbol{\psi}_0$$
$$\mathbf{n}=-\left(\frac{\partial H_0}{\partial \mathbf{I}^T}\right)\bigg|_{\mathbf{I}=\mathbf{I}_0}\equiv-\frac{\partial H_0}{\partial \mathbf{I}_0^T}=-\left(\frac{\partial H_0}{\partial \mathbf{I}_0}\right)^T$$

Here $\mathbf{I}_0, \mathbf{J}_0, \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0$ are arbitrary constants of integration, and $\mathbf{n}$ is a column vector of the frequencies of the unperturbed solution.

Let us suppose that $(p_1^{(0)},\ldots,p_l^{(0)})^T=\mathbf{a}_1$ are found such that the corresponding frequencies will be rationally commensurable with Ω, i.e.

$$(\mathbf{n}^{(0)}(\mathbf{a}_1),\Omega)^T=c(\bar{k}_1,\ldots,\bar{k}_l,\bar{k}_{l+1})^T,\quad \mathbf{n}^{(0)}=-\partial H_0/\partial \mathbf{a}_1^T \tag{1.4}$$

Here $c\neq 0$ is an arbitrary constant $\bar{k}_i\in Z$ and LCM $(\bar{k}_1,\ldots,\bar{k}_l,\ \bar{k}_{l+1})=1$. Then we shall call the solution of the unperturbed system corresponding to $\mathbf{a}_1$ according to Poincaré, periodic with period $T=2\pi\bar{k}_{N+1}/\Omega$. The solution (or, more accurately, a family of periodic solutions), is given by the formulas

$$\mathbf{I}=\mathbf{a}_1,\quad \mathbf{J}=\mathbf{J}_0,\quad \boldsymbol{\varphi}=\mathbf{n}^{(0)}t+\boldsymbol{\varphi}_0,\quad \boldsymbol{\psi}=\boldsymbol{\psi}_0 \tag{1.5}$$

We shall seek the T-periodic solutions of the initial system (1.1)-(1.3) which will transform to the solution (1.5) when $\mu=0$. It is possible that such solutions exist only for some specified values of $\mathbf{J}_0, \boldsymbol{\varphi}_0$ and $\boldsymbol{\psi}_0$. We shall call them the generating solutions and denote them by $\mathbf{a}_2$, $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ respectively. We shall also call the periodic solution of the unperturbed system

$$\mathbf{I}=\mathbf{a}_1,\quad \mathbf{J}=\mathbf{a}_2,\quad \boldsymbol{\varphi}=\mathbf{n}^{(0)}t+\boldsymbol{\omega}_1,\quad \boldsymbol{\psi}=\boldsymbol{\omega}_2 \tag{1.6}$$

to which the T-periodic solution of the system (1.1)-(1.3) tends as $\mu\to 0$, the generating solution.

Let us consider the general solution of system (1.1)-(1.3) using Poincaré theorem /1/.

Theorem 1. Let the following system be given:

$$\frac{d\mathbf{x}}{dt}=\mathbf{X}(\mathbf{x},t,\mu)\equiv\mathbf{X}^{(0)}(\mathbf{x},t)+\mu\mathbf{X}^{(1)}(\mathbf{x},t)+\mu^2\mathbf{X}^{(2)}(\mathbf{x},t)+\ldots \tag{1.7}$$
$$\mathbf{x}=(x_1,\ldots,x_N)^T,\quad \mathbf{X}(\mathbf{x},t,\mu)=(X_1,X_2,\ldots,X_N)^T$$

Then, provided that: 1) when $\mu=0$, the system (1.7) has a solution $\mathbf{x}=\mathbf{x}^{(0)}(t)$ analytic when $|t-t_0|\leqslant h$; 2) the function $\mathbf{X}(\mathbf{x},t,\mu)$ is analytic in the region $G\times(t_0-h,\ t_0+h)\times(-\mu_0,\ \mu_0)$ where G denotes the neighbourhood of the set $\{\mathbf{x}\in R^N\colon \mathbf{x}=\mathbf{x}^{(0)}(t),\ |t-t_0|\leqslant h\}$; 3) the parameters μ and $\nu_s\ (s=1,\ldots,N)$ are sufficiently small in modulus, where $\mu\in(-\mu_0,\ \mu_0)$, $\nu_s=x_s(t_0)-x_s^{(0)}(t_0)$, $x_s(t_0)$ and $x_s^{(0)}(t_0)$ are the initial values of the variables for the solution sought and for the generating solution, then when $\mu\neq 0$, $t\in(t_0-h,\ t_0+h)$, then the solution of system (1.7) will be represented by the following, absolutely convergent series:

$$x_s(t) = x_s^{(0)}(t) + \sum_{\|\mathbf{m}\|+m_{N+1}\geqslant 1} x_s^{(\mathbf{m},\, m_{N+1})}(t)\, \nu_1^{m_1} \ldots \nu_N^{m_N} \mu^{m_{N+1}} \tag{1.8}$$

$$\mathbf{m} = (m_1, \ldots, m_N),\ m_i = 0, 1, 2, \ldots\ (i = 1, 2, \ldots, N+1)$$

In particular, when $x_s(t_0) = x_s^{(0)}(t_0)$, then the solution of (1.7) will have the form

$$x_s(t) = x_s^{(0)}(t) + \sum_{n=1}^{\infty} \mu^n x_s^{(n)}(t) \tag{1.9}$$

The results of the above theorem can be extended at once to the system of canonical Eqs. (1.1)-(1.3).

Therefore the solution of (1.1)-(1.3) with arbitrary initial conditions

$$t = 0,\quad \mathbf{I} = \mathbf{I}_0,\quad \mathbf{J} = \mathbf{J}_0,\quad \boldsymbol{\varphi} = \boldsymbol{\varphi}_0,\quad \boldsymbol{\psi} = \boldsymbol{\psi}_0 \tag{1.10}$$

i.e. the general solution for sufficiently small μ, is represented by the absolutely convergent series of the form (1.9)

$$\mathbf{U} = \mathbf{U}_0 + \sum_{m=1}^{\infty} \mu^m \mathbf{U}_m(\mathbf{I}_0, \mathbf{J}_0, \mathbf{n}t + \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0, t) \tag{1.11}$$

$$\boldsymbol{\varphi} = \mathbf{n}t + \boldsymbol{\varphi}_0 + \sum_{m=1}^{\infty} \mu^m \boldsymbol{\varphi}_m(\mathbf{I}_0, \mathbf{J}_0, \mathbf{n}t + \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0, t)$$

$$\mathbf{U}^T = (\mathbf{I}^T, \mathbf{J}^T, -\boldsymbol{\psi}^T),\quad \mathbf{U}_s^T = (\mathbf{I}_s^T, \mathbf{J}_s^T, -\boldsymbol{\psi}_s^T)\quad (s = 0, 1, 2, \ldots)$$

We note that by virtue of the choice of initial conditions we have

$$t = 0,\quad \mathbf{I}_m = \boldsymbol{\varphi}_m = 0,\quad \mathbf{J}_m = \boldsymbol{\psi}_m = 0\quad (m = 1, 2, \ldots)$$

The series (1.11) are convergent in the time interval $t \in (-h, h)$ (for sufficiently low values of μ, $h > T$) and their coefficients are found by integrating a known sequence of systems of differential equations.

2. A method of studying the conditions of periodicity of the solutions.

The solution (1.1) will be T-periodic if and only if the following conditions hold:

$$\mathbf{U}(T) - \mathbf{U}(0) = \mathbf{0},\quad \boldsymbol{\varphi}(T) - \boldsymbol{\varphi}(0) - \mathbf{n}^{(0)}T = \mathbf{0} \tag{2.1}$$

We obtain the solution of Eqs. (1.1)-(1.3) with initial conditions

$$\mathbf{I}_0 = \mathbf{a}_1 + \boldsymbol{\beta}_1,\quad \mathbf{J}_0 = \mathbf{a}_2 + \boldsymbol{\beta}_2,\quad \boldsymbol{\varphi}_0 = \boldsymbol{\omega}_1 + \boldsymbol{\gamma}_1,\quad \boldsymbol{\psi}_0 = \boldsymbol{\omega}_2 + \boldsymbol{\gamma}_2 \tag{2.2}$$

where $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ are small quantities, directly from formulas (1.11). The solution will be represented by the series

$$\mathbf{U} = (\mathbf{a}_1 + \boldsymbol{\beta}_1, \mathbf{a}_2 + \boldsymbol{\beta}_2, -\boldsymbol{\omega}_2 - \boldsymbol{\gamma}_2)^T + \sum_{m=1} \mu^m \mathbf{U}_m(\mathbf{a}_1 + \boldsymbol{\beta}_1, \mathbf{a}_2 + \boldsymbol{\beta}_2, \mathbf{n}t + \boldsymbol{\omega}_1 + \boldsymbol{\gamma}_1, \boldsymbol{\omega}_2 + \boldsymbol{\gamma}_2, t) \tag{2.3}$$

$$\boldsymbol{\varphi} = \mathbf{n}t + \boldsymbol{\omega}_1 + \boldsymbol{\gamma}_1 + \sum_{m=1} \mu^m \boldsymbol{\varphi}_m(\mathbf{a}_1 + \boldsymbol{\beta}_1, \mathbf{a}_2 + \boldsymbol{\beta}_2, \mathbf{n}t + \boldsymbol{\omega}_1 + \boldsymbol{\gamma}_1, \boldsymbol{\omega}_2 + \boldsymbol{\gamma}_2, t)$$

The quantities $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ have the same meaning as the parameters ν_s in Poincaré's theorem. This means that we can arrive at a series of the form (1.8) also by expanding the coefficients $\mathbf{I}_m, \mathbf{J}_m, \boldsymbol{\varphi}_m, \boldsymbol{\psi}_m$ of the solution (2.3) in series in powers of $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$. To find the coefficients of the series (2.3) it is sufficient, first to construct a solution of (1.1)-(1.3) in the form of the series (1.11), and then replace the initial conditions in them according to formulas (2.2).

Substituting series (2.3) into the conditions of periodicity (2.1), we obtain

$$\boldsymbol{\Psi}_1(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \mu) \equiv \mathbf{U}(T) - \mathbf{U}(0) = \mu\mathbf{U}_1(T) + \mu^2\mathbf{U}_2(T) + \ldots = \mathbf{0} \tag{2.4}$$

$$\boldsymbol{\Psi}_2(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \mu) \equiv \boldsymbol{\varphi}(T) - \boldsymbol{\varphi}(0) - \mathbf{n}^{(0)}T = (\mathbf{n} - \mathbf{n}^{(0)})T + \mu\boldsymbol{\varphi}_1(T) + \ldots = \mathbf{0}$$

Next we construct Eqs. (2.4) in explicit form (with the necessary accuracy of up to $\boldsymbol{\beta}, \boldsymbol{\gamma}, \mu$) and apply the theorem on implicit functions.

In order to construct Eqs. (2.4), we must determine the coefficients of the series (2.3) and their values at the instant of time $t = T$. The coefficients will be known, if we construct the solution in the form (1.11). The coefficients of series (1.11) in turn are determined as a result of integrating a known sequence of systems of differential equations (we omit these equations for brevity). The computation of the functions $\mathbf{I}_m(t)$, $\mathbf{J}_m(t)$, $\boldsymbol{\varphi}_m(t)$, $\boldsymbol{\psi}_m(t)$ $(m = 1, 2, \ldots)$ is reduced to quadratures.

Further, using the coefficients $\mathbf{I}_m, \mathbf{J}_m, \boldsymbol{\varphi}_m, \boldsymbol{\psi}_m$ computed up to the prescribed order and the substitution (2.2), we construct the conditions of periodicity of (2.4) in the form of series in powers of the small quantities $\boldsymbol{\beta}, \boldsymbol{\gamma}, \mu$ to the required accuracy.

Thus the method of investigating the conditions of periodicity consists of the following sequence of operations;

1°. We construct the first terms of the series (1.11) representing the general solution of Eqs.(1.1)-(1.3).

2°. We find the values of the coefficients of these series for the instant of time $t = T$.

3°. Using the substitution (2.2) we write the conditions of perodicity in the from (2.4) and carry out the necessary expansions in powers of the small parameters β_i, γ_i, μ.

4°. Using the corresponding theorems on implicit functions, we derive analytic formulas representing the conditions of solvability of Eqs.(2.4).

We note that Proskuryakov used the same method when studying the periodic solutions of quasilinear systems /2/. Unlike in /2/, the present method is developed for the non-linear systems (1.1)-(1.3) whose frequencies depend on the amplitudes.

The method can be made very flexible and efficient by using, at various stages of the investigation of periodicity, not all the substitutions (2.2) simultaneously, but only some of them.

We can obtain the following representations for the fundamental coefficients of (2.4):

$$\frac{\mathbf{U}_1(T)}{T} = \left(\frac{\partial^2 \langle H_1\rangle}{\partial \mathbf{a}_1 \partial \mathbf{y}_0{}^T} - \frac{T}{2}\frac{\partial^2 \langle H_1\rangle}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0{}^T}\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T} - \left\{\int \frac{\partial^2 \{H_1\}}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0{}^T} dt\right\}_{t=0} \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\right)\boldsymbol{\beta}_1 + \frac{\partial \langle H_1\rangle}{\partial \mathbf{y}_0{}^T} + O(\boldsymbol{\beta}_1^2) \tag{2.5}$$

$$\frac{\boldsymbol{\varphi}_1(T)}{T} = -\frac{T}{2}\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\frac{\partial \langle H_1\rangle}{\partial \boldsymbol{\varphi}_0{}^T} + \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\left\{\int \frac{\partial \{H_1\}}{\partial \boldsymbol{\varphi}_0{}^T} dt\right\}_{t=0} - \frac{\partial \langle H_1\rangle}{\partial \mathbf{a}_1{}^T} + O(\boldsymbol{\beta}_1)$$

$$\frac{\mathbf{U}_2(T)}{T} = \frac{T}{2}\frac{\partial \langle \mathbf{V}\rangle}{\partial \mathbf{y}_0{}^T}\langle \mathbf{W}\rangle - \frac{\partial \langle \mathbf{V}\rangle}{\partial \mathbf{y}_0{}^T}\left\{\int \{\mathbf{W}\}\, dt\right\}_{t=0} + \left\{\int \frac{\partial \{\mathbf{V}\}}{\partial \mathbf{y}_0{}^T} dt\right\}_{t=0}\langle \mathbf{W}\rangle - \left(\frac{T^2}{6}\frac{\partial^2 \langle H_1\rangle}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0{}^T} + \frac{T}{2}\left\{\int \frac{\partial^2 \{H_1\}}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0{}^T} dt\right\}_{t=0} - \left\{\int \left\{\int \frac{\partial^2 \{H_1\}}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0{}^T} dt\right\} dt\right\}_{t=0}\right)\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\frac{\partial \langle H_1\rangle}{\partial \boldsymbol{\varphi}_0{}^T} + \left\langle \frac{\partial \{\mathbf{V}\}}{\partial \mathbf{y}_0{}^T}\left\{\int \{\mathbf{W}\}\, dt\right\}\right\rangle + \frac{\partial \langle H_2\rangle}{\partial \mathbf{y}_0{}^T} + O(\boldsymbol{\beta}_1)$$

Here

$$\mathbf{W}^T \equiv \left(\frac{\partial H_1}{\partial \boldsymbol{\varphi}_0{}^T}, \frac{\partial H_1}{\partial \boldsymbol{\psi}_0{}^T}, -\frac{\partial H_1}{\partial \mathbf{J}_0{}^T}, -\frac{\partial H_1}{\partial \mathbf{a}_1{}^T} - \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\int_0^t \frac{\partial H_1}{\partial \boldsymbol{\varphi}_0{}^T} dt\right)$$

$$\mathbf{V} \equiv \left(\frac{\partial H_1}{\partial \mathbf{a}_1}, \frac{\partial H_1}{\partial \mathbf{J}_0}; \frac{\partial H_1}{\partial \boldsymbol{\psi}_0}, \frac{\partial H_1}{\partial \boldsymbol{\varphi}_0}\right), \qquad \mathbf{y}_0{}^T = (\boldsymbol{\varphi}_0{}^T, \boldsymbol{\psi}_0{}^T, \mathbf{J}_0{}^T)$$

In (2.5) and further formulas in this paper we represent the periodic functions $f = f(\mathbf{a}_1, \mathbf{J}_0, \mathbf{n}^{(0)}t + \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0, t)$ by $f = \langle f\rangle + \{f\}$, where

$$\langle f\rangle = \frac{1}{T}\int_0^T f(t)\, dt$$

is a constant component of the function f and $\{f\}$ is its purely periodic part.

3. Classical conditions of existence.

Expanding the coefficients of the first approximation in (2.5) in series in powers of the small quantities $\boldsymbol{\beta}_i, \boldsymbol{\gamma}_i$, we write the conditions of periodicity in the form

$$\frac{\boldsymbol{\Psi}_1}{\mu T} \equiv \frac{\partial \langle H_1\rangle}{\partial \boldsymbol{\xi}_0{}^T} + \frac{\partial^2 \langle H_1\rangle}{\partial \mathbf{a}_2 \partial \boldsymbol{\xi}_0{}^T}\boldsymbol{\beta}_2 + \frac{\partial^2 \langle H_1\rangle}{\partial \boldsymbol{\omega}_1 \partial \boldsymbol{\xi}_0{}^T}\boldsymbol{\gamma}_1 + \frac{\partial^2 \langle H_1\rangle}{\partial \boldsymbol{\omega}_2 \partial \boldsymbol{\xi}_0{}^T}\boldsymbol{\gamma}_2 + (\ldots)\boldsymbol{\beta}_1 + \ldots + \mu(\ldots) = 0 \tag{3.1}$$

$$\boldsymbol{\Psi}_2 \equiv -T\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1{}^T}\boldsymbol{\beta}_1 + (\ldots)\boldsymbol{\beta}_1\boldsymbol{\beta}_1 + \mu(\ldots) = 0;$$

$$\boldsymbol{\xi}_0{}^T = (\boldsymbol{\omega}_1{}^T, \boldsymbol{\omega}_2{}^T, \mathbf{a}_2{}^T)$$

We can determine $\beta_1, \beta_2, \gamma_1, \gamma_2$ from system (3.1) as holomorphic functions of the parameter μ, provided that the equations contain no free terms and the determinant of the coefficients accompanying the unknowns is different from zero. As a result we obtain the classical conditions of Poincaré.

Theorem 2. Eqs.(1.1)-(1.3) have periodic solutions with period $T = 2\pi k_{N+1}/\Omega$, generated from the solution (1.6), if the parameters $\mathbf{a}$ and $\boldsymbol{\omega}$ of this solution satisfy the conditions

$$(\mathbf{n}^{(\omega)T}(\mathbf{a}_1), \Omega) = c(\bar{k}_1, \ldots, \bar{k}_l, \bar{k}_{N+1}), \quad c \neq 0, \quad \bar{k}_i \in Z \tag{3.2}$$

$$\frac{\partial \langle H_1 \rangle}{\partial \boldsymbol{\omega}_1^T} = 0, \quad \frac{\partial \langle H_1 \rangle}{\partial \mathbf{a}_2^T} = \frac{\partial \langle H_1 \rangle}{\partial \boldsymbol{\omega}_2^T} = 0 \tag{3.3}$$

$$\det \left\| \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \right\| \neq 0 \tag{3.4}$$

$$\det \left\| \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\xi}_0 \partial \boldsymbol{\xi}_0^T} \right\| \equiv \begin{vmatrix} \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_1 \partial \boldsymbol{\omega}_1^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_2 \partial \boldsymbol{\omega}_1^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \mathbf{a}_2 \partial \boldsymbol{\omega}_1^T} \\ \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_1 \partial \boldsymbol{\omega}_2^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_2 \partial \boldsymbol{\omega}_2^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \mathbf{a}_2 \partial \boldsymbol{\omega}_2^T} \\ \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_1 \partial \mathbf{a}_2^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \boldsymbol{\omega}_2 \partial \mathbf{a}_2^T} & \frac{\partial^2 \langle H_1 \rangle}{\partial \mathbf{a}_2 \partial \mathbf{a}_2^T} \end{vmatrix} \neq 0 \tag{3.5}$$

The periodic solutions are represented by series in integral powers of the parameter μ, converging for all values of time t, provided that the value of μ is sufficiently small.

4. Conditions for the existence of periodic solutions in some special cases.

In a number of problems of celestial mechanics the function $\langle H_1 \rangle$ is independent, for the specified commensurabilities of the frequencies (1.4), of the generating values of the variables $\mathbf{J}_0, \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0$, i.e.

$$\langle H_1(\mathbf{a}_1, \mathbf{J}_0, \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0) \rangle \equiv \langle H_1(\mathbf{a}_1) \rangle \tag{4.1}$$

(in particular, the function $\langle H_1 \rangle$ may be identically equal to zero (see e.g. /4/) or depend on some of the quantities only). In this case conditions (3.3) are satisfied identically, but condition (3.5) does not hold. This, naturally, does not imply that there are no periodic solutions for the commensurabilities mentioned above. In this case we must obtain new sufficient conditions of existence, and we derive them below.

Using formulas (2.5) and taking condition (4.1) into account, we can write the necessary and sufficient conditions for the existence of periodic solutions of (2.4) as follows:

$$\frac{\boldsymbol{\Psi}_1}{\mu T} \equiv -\left\{\int \frac{\partial^2 \{H_1\}}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0^T} dt\right\}_{t=0} \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \boldsymbol{\beta}_1 + \mu \left(\left\langle \frac{\partial \{\mathbf{V}\}}{\partial \mathbf{y}_0^T} \left\{\int \{\mathbf{W}\}\, dt\right\}\right\rangle + \left\{\int \frac{\partial^2 \{H_1\}}{\partial \boldsymbol{\varphi}_0 \partial \mathbf{y}_0^T} dt\right\}_{t=0} \left(\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \left\{\int \frac{\partial \{H_1\}}{\partial \boldsymbol{\varphi}_0^T} dt\right\}_{t=0} - \frac{\partial \langle H_1 \rangle}{\partial \mathbf{a}_1^T}\right) + \frac{\partial \langle H_2 \rangle}{\partial \mathbf{y}_0^T}\right) + \ldots = 0 \tag{4.2}$$

$$-\frac{\boldsymbol{\Psi}_2}{T} \equiv \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \boldsymbol{\beta}_1 + \mu \left(\frac{\partial \langle H_1 \rangle}{\partial \mathbf{a}_1^T} - \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \times \left\{\int \frac{\partial \{H_1\}}{\partial \boldsymbol{\varphi}_0^T} dt\right\}_{t=0}\right) + \ldots = 0$$

Let condition (3.4) hold. Then the second equation of (4.2) will yield

$$\boldsymbol{\beta}_1 = \mu \left(\left\{\int \frac{\partial \{H_1\}}{\partial \boldsymbol{\varphi}_0^T} dt\right\}_{t=0} - \left(\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T}\right)^{-1} \frac{\partial \langle H_1 \rangle}{\partial \mathbf{a}_1^T}\right) + \mu^2(\ldots)$$

Substituting this result into the first equation of (4.2) and carrying out the necessary reduction, we obtain

$$\frac{\boldsymbol{\Psi}_1}{\mu^2 T} \equiv \left\langle \frac{\partial \{\mathbf{V}\}}{\partial \mathbf{y}_0^T} \left\{\int \{\mathbf{W}\}\, dt\right\}\right\rangle + \frac{\partial \langle H_2 \rangle}{\partial \mathbf{y}_0^T} + \mu(\ldots) = 0 \tag{4.3}$$

Using conditions (2.3), we expand the left-hand sides of Eqs.(4.3) in series in powers of $\boldsymbol{\beta}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$

$$\frac{\boldsymbol{\Psi}_1}{\mu^2 T} \equiv \langle \boldsymbol{\Phi} \rangle + \frac{\partial \langle H_2 \rangle}{\partial \boldsymbol{\xi}_0^T} + \left(\frac{\partial \langle \boldsymbol{\Phi} \rangle}{\partial \mathbf{a}_2} + \frac{\partial^2 \langle H_2 \rangle}{\partial \mathbf{a}_2 \partial \boldsymbol{\xi}_0^T}\right) \boldsymbol{\beta}_2 + \left(\frac{\partial \langle \boldsymbol{\Phi} \rangle}{\partial \boldsymbol{\omega}_1} + \frac{\partial^2 \langle H_2 \rangle}{\partial \boldsymbol{\omega}_1 \partial \boldsymbol{\xi}_0^T}\right) \boldsymbol{\gamma}_1 + \left(\frac{\partial \langle \boldsymbol{\Phi} \rangle}{\partial \boldsymbol{\omega}_2} + \frac{\partial^2 \langle H_2 \rangle}{\partial \boldsymbol{\omega}_2 \partial \boldsymbol{\xi}_0^T}\right) \boldsymbol{\gamma}_2 + \ldots = 0 \tag{4.4}$$

Here

$$\boldsymbol{\Phi} \equiv \left(\frac{\partial \{\mathbf{V}\}}{\partial \mathbf{y}_0^T} \left\{\int \{\mathbf{W}\}\, dt\right\}\right)\Big|_{\mathbf{y}_0 = \boldsymbol{\xi}_0} \equiv \frac{\partial^2 \{H_1\}}{\partial \mathbf{a}_1 \partial \boldsymbol{\xi}_0^T} \left\{\int \frac{\partial \{H_1\}}{\partial \boldsymbol{\omega}_1^T} dt\right\} + \tag{4.5}$$

$$\frac{\partial^2\{H_1\}}{\partial \mathbf{a}_2 \partial \xi_0^T}\left\{\int \frac{\partial \{H_1\}}{\partial \omega_2^T}\,dt\right\} - \frac{\partial^2\{H_1\}}{\partial \omega_2 \partial \xi_0^T}\left\{\int \frac{\partial \{H_1\}}{\partial \mathbf{a}_2^T}\,dt\right\} -$$
$$\frac{\partial^2\{H_1\}}{\partial \omega_1 \partial \xi_0^T}\left\{\int \frac{\partial \{H_1\}}{\partial \mathbf{a}_1^T}\,dt\right\} - \frac{\partial^2\{H_1\}}{\partial \omega_1 \partial \xi_0^T}\,\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T}\left\{\int\left\{\int \frac{\partial \{H_1\}}{\partial \omega_1^T}\,dt\right\}dt\right\}$$

$$\langle H_2\rangle = \frac{1}{T}\int_0^T H_2(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}^{(0)}t + \omega_1, \omega_2, t)\,dt$$

$$\langle \Phi\rangle = \frac{1}{T}\int_0^T \Phi(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}^{(0)}t + \omega_1, \omega_2, t)\,dt$$

Now using the theorem on implicit functions, we shall formulate the conditions for the existence of periodic solutions for the case in question, in the form of the following theorem.

Theorem 3. When (4.1) is degenerate, Eqs.(1.1)-(1.3) have T-periodic solutions generated from solution (1.6), provided that the parameters $\mathbf{a}$ and ω of this solution satisfy the conditions

$$(\mathbf{n}^{(0)}(\mathbf{a}_1), \Omega) = c(\bar{k}_1, \ldots, \bar{k}_l, \bar{k}_{N+1})^T \tag{4.6}$$
$$\langle \Phi\rangle + \frac{\partial\langle H_2\rangle}{\partial \xi_0^T} = 0, \quad \det\left\|\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T}\right\| \neq 0,$$
$$\det\left\|\frac{\partial\langle \Phi\rangle}{\partial \xi_0} + \frac{\partial^2\langle H_2\rangle}{\partial \xi_0 \partial \xi_0^T}\right\| \neq 0$$

When conditions (4.6) hold, the system of Eqs.(4.2) yields uniquely the quantities $\beta_1, \beta_2, \gamma_1, \gamma_2$ as holomorphic functions of the parameter μ. The periodic solutions are represented by series in integral powers of μ, converging at low values of this parameter over the whole time interval $t \in (-\infty, +\infty)$.

In exactly the same manner we consider a more general special case when the function $\langle H_1\rangle$ depends only on some of the quantities appearing in $\mathbf{J}_0, \varphi_0, \psi_0$, i.e.

$$\langle H_1\rangle \equiv \langle H_1(\mathbf{J}_0^*, \varphi_0^*, \psi_0^*)\rangle \tag{4.7}$$
$$\mathbf{J}_0^* = (p_{l+1}^{(0)}, \ldots, p_{l+s_1}^{(0)})^T, \quad \varphi_0^* = (q_1^{(0)}, \ldots, q_{s_1}^{(0)})^T, \quad \psi_0^* = (q_{l+1}^{(0)}, \ldots, q_{l+s_1}^{(0)})^T$$
$$(p_{l+s_1+1}, \ldots, p_N)^T = \mathbf{J}^{**}, \quad (q_{s_1+1}, \ldots, q_l) = \varphi^{**},$$
$$(q_{l+s_1+1}, \ldots, q_N)^T = \psi^{**}$$
$$\mathbf{J} = (\mathbf{J}^*, \mathbf{J}^{**})^T, \quad \varphi = (\varphi^*, \varphi^{**})^T, \quad \psi = (\psi_1^*, \psi^{**})^T$$

We denote the corresponding generating values of the variables introduced above, as follows:

$$\mathbf{J}^* = \mathbf{a}_{21}, \quad \mathbf{J}^{**} = \mathbf{a}_{22}, \quad \varphi^* = \omega_{11}, \quad \varphi^{**} = \omega_{12}, \quad \psi^* = \omega_{21}, \quad \psi^{**} = \omega_{22}$$
$$\xi_{01}^T = (\omega_{11}^T, \omega_{21}^T, \mathbf{a}_{21}^T), \quad \xi_{02}^T = (\omega_{12}^T, \omega_{22}^T, \mathbf{a}_{22}^T)$$

The method of Sect.2 are used to ascertain the existence of periodic solutions in the present case. Omitting for brevity the analysis of the conditions of periodicity (2.4), we formulate the final result as the following theorem.

Theorem 4. In the special case (4.7) Eqs.(1.1)-(1.3) admit T-periodic solutions generated from solution (1.6) whose parameters $\mathbf{a}$ and ω satisfy the conditions

$$(\mathbf{n}^{(0)}(\mathbf{a}_1), \Omega)^T = c(k_1, \ldots, k_l, k_{N+1})^T$$
$$\frac{\partial\langle H_1\rangle}{\partial \xi_{01}^T} \equiv \left(\frac{\partial\langle H_1\rangle}{\partial \omega_{11}^T}, \frac{\partial\langle H_1\rangle}{\partial \omega_{21}^T}, \frac{\partial\langle H_1\rangle}{\partial \mathbf{a}_{21}^T}\right)^T = 0, \quad \langle \Phi^{**}\rangle + \frac{\partial\langle H_2\rangle}{\partial \xi_{02}^T} = 0$$
$$\det\left\|\frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T}\right\| \neq 0, \quad \det\left\|\frac{\partial\langle H_1\rangle}{\partial \xi_{01} \partial \xi_{01}^T}\right\| \neq 0,$$
$$\det\left\|\frac{\partial\langle \Phi^{**}\rangle}{\partial \xi_{02}} + \frac{\partial^2\langle H_2\rangle}{\partial \xi_{02} \partial \xi_{02}^T}\right\| \neq 0 \;\cdot$$
$$\left(\Phi^{**} = \left(\frac{\partial\{\mathbf{V}\}}{\partial \mathbf{y}_{02}^T}\left\{\int\{\mathbf{W}\}\,dt\right\}\right)\Big|_{\mathbf{y}_0 = \xi_0}, \quad \mathbf{y}_{02} = (\varphi_0^{**}, \psi_0^{**}, \mathbf{J}_0^{**})^T\right)$$

5. Periodic solutions, degenerate in the $(S-1)$-th approximation. Within the specified commensurability of the unperturbed frequencies (1.4) we may encounter the case in which the conditions of periodicity of the solutions of the equations for the first $S-1$-approximations $\mathbf{I}_i, \mathbf{J}_i, \psi_i$ $(i = 1, \ldots, S-1)$ will be satisfied identically, and the resonant terms will appear in condition (2.4) only for $\mathbf{I}_S, \mathbf{J}_S, \psi_S$. We shall call the periodic solutions with the above property provisionally degenerate in the $(S-1)$-th approximation.

The conditions for the existence of the periodic solutions in question are obtained as a result of studying the terms of (2.4) up to the order μ^S, with help of the methods of Sect.2.

Omitting the technical details of constructing these conditions, we formulate the final result as follows.

Theorem 5. Eqs.(1.1)-(1.3) have T-periodic solutions, degenerate in the $(S-1)$-th approximation, generated from (1.6), provided that the quantities $\mathbf{a}$ and $\boldsymbol{\omega}$ satisfy the conditions

$$(\mathbf{n}^{(0)}(\mathbf{a}_1), \Omega)^T = c(k_1, \ldots, k_l, k_{N+1})^T$$
$$\langle \boldsymbol{\Phi}^{(S-1)} \rangle + \frac{\partial \langle H_S \rangle}{\partial \boldsymbol{\xi}_0^T} = 0$$
$$\det \left\| \frac{\partial^2 H_0}{\partial \mathbf{a}_1 \partial \mathbf{a}_1^T} \right\| \neq 0, \quad \det \left\| \frac{\partial \langle \boldsymbol{\Phi}^{(S-1)} \rangle}{\partial \boldsymbol{\xi}_0} + \frac{\partial^2 \langle H_S \rangle}{\partial \boldsymbol{\xi}_0 \partial \boldsymbol{\xi}_0^T} \right\| \neq 0$$

where the function $\boldsymbol{\Phi}^{(S-1)}$ generalizing the function (4.5) is given by the formula

$$\boldsymbol{\Phi}^{(S-1)}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}^{(0)}t + \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, t) \equiv \tag{5.1}$$
$$\sum_{j=1}^{S-1} \sum_{m=1}^{S-j} \frac{1}{m!} \sum_{\alpha_I + \alpha_J + \alpha_\varphi + \alpha_\psi = m} \frac{\partial^{m+1} H_j}{\partial \mathbf{I}_0^{\alpha_I} \partial \mathbf{J}_0^{\alpha_J} \partial \boldsymbol{\varphi}_0^{\alpha_\varphi} \partial \boldsymbol{\psi}_0^{\alpha_\psi} \partial \mathbf{y}_0^T} \times$$
$$\sum{}' \{\boldsymbol{\psi}_{r_1}\}^{K_1^\psi} \ldots \{\boldsymbol{\psi}_{r_m}\}^{K_m^\psi} \{\boldsymbol{\varphi}_{r_1}\}^{K_1^\varphi} \ldots \{\boldsymbol{\varphi}_{r_m}\}^{K_m^\varphi} \{\mathbf{J}_{r_1}\}^{K_1^J} \ldots$$
$$\{\mathbf{J}_{r_m}\}^{K_m^J} \{\mathbf{I}_{r_1}\}^{K_1^I} \ldots \{\mathbf{I}_{r_m}\}^{K_m^I} \Big|_{\substack{\mathbf{I}_0 = \mathbf{a}_1 \\ \mathbf{y}_0 = \boldsymbol{\xi}_1}}$$

Here $\sum'$ denotes summation over the values of $K_1^\psi, \ldots, K_m^\psi, K_1^\varphi, \ldots, K_m^I, r_1, \ldots, r_m$ for which the conditions $r_1 + \ldots + r_m = S - j$, $K_1^{(I,J,\varphi,\psi)} + \ldots + K_m^{(I,J,\varphi,\psi)} = \alpha_{(I,J,\varphi,\psi)}$ hold. All summation indices in (5.1) take integral non-negative values and the derivatives are computed in the operator sense. When $r < S$, the functions $\boldsymbol{\psi}_r(\mathbf{I}_0, \mathbf{J}_0, \mathbf{n}^{(0)}t + \boldsymbol{\varphi}_0, \boldsymbol{\psi}_0, t), \ldots, \mathbf{I}_r, \ldots$ are purely periodic functions of time t, and can be determined uniquely using the method of successive approximations.

Many more special cases can be studied in exactly the same manner, e.g. when degeneration occurs with respect to some of the variables in all the r $(r < S)$ approximations. The results obtained here can be generalized to cover autonomous Hamiltonian systems.

6. Appendix. We shall use the results obtained to investigate the periodic motions of a heavy rigid body about a fixed point. We assume that, dynamically, the body is nearly axisymmetric and its centre of mass lies near a fixed point.

Retaining the notation of /3/, we write the equations of motion of the rigid body in the form of a canonical, single power system with Hamiltonian K

$$K(L, l, g) = K_0(L) + \sum_{\sigma=1}^{\infty} \mu^\sigma K_\sigma(L, l, g) \tag{6.1}$$
$$K_0 = \sqrt{-2c_1 - L^2(\varkappa - 1)}, \quad K_1 = -\frac{F_1(L, G, l, g)}{K_0(L)}\Big|_{G=K_0}$$
$$K_2 = -\frac{K_1^2}{2K_0} + \frac{F_1}{K_0^2}\frac{\partial F_1}{\partial G}\Big|_{G=K_0}; \quad \mu = \frac{A-B}{A}, \quad \varkappa = \frac{A}{C}$$

Here μ is a small parameter, A, B, C are the moments of inertia of the body, c_1 is the energy constant, g plays the part of the independent variable and F_1 is represented by a known function of the Andoyer variables L, G, H, l, g:

$$F_1 = f_{0.0} + f_{0.1}\cos g + f_{2.0}\cos 2l + f_{1.0}\cos(l + \lambda) + f_{1.1}\cos(l + g + \lambda) + f_{1.-1}\cos(l - g + \lambda)$$

The coefficients f_{ij} are determined, using the formulas given in /3/, as functions of the variables G, θ, ρ $(\cos\theta = L/G, \cos\rho = H/G)$.

When $\mu = 0$, the periodic solution of the problem has the form

$$L = a, \quad l = ng + \omega, \quad n = -\frac{\partial K_0}{\partial a} = \frac{n_l}{n_g} = \frac{q_2}{q_1} \quad (q_1, q_2 \in Z) \tag{6.2}$$
$$n_l = a(\varkappa - 1), \quad n_g = \sqrt{-2c_1 - a^2(\varkappa - 1)}$$

Here a satisfies the condition of commensurability of the unperturbed frequencies n_l and n_g, ω is an arbitrary constant.

Using Poincaré's conditions (3.2)-(3.5) the existence of periodic solutions of the problem in question was proved in /4/ for the case of commensurability $n_l = \pm n_g$.

Using the results of Theorem 3, we shall show the existence of periodic solutions of problem (6.1), generated, in the case of the commensurabilities 1) $n_l = \pm 2n_g$; 2) $3n_l = \pm n_g$;

3) $2n_l = \pm n_g$ from (6.2).

The quantities a, ω must satisfy, for these solutions, the conditions

$$\langle\Phi\rangle + \frac{\partial\langle K_2\rangle}{\partial\omega} = 0, \quad \frac{\partial^2 K_0}{\partial a^2} \neq 0, \quad \frac{\partial\langle\Phi\rangle}{\partial\omega} + \frac{\partial^2\langle K_2\rangle}{\partial\omega^2} \neq 0, \quad q_1 n_l = q_2 n_g \tag{6.3}$$

$$\langle\Phi\rangle \equiv \left\langle \frac{\partial^2 K_1}{\partial a\,\partial\omega} \left\{\int \frac{\partial K_1}{\partial\omega}\, dt\right\}\right\rangle - \left\langle \frac{\partial^2 K_1}{\partial\omega^2} \left\{\int \frac{\partial K_1}{\partial a}\, dt\right\}\right\rangle -$$
$$\left\langle \frac{\partial^2 K_1}{\partial\omega^2} \frac{\partial^2 K_0}{\partial a^2} \left\{\int \left\{\int \frac{\partial K_1}{\partial\omega}\, dt\right\} dt\right\}\right\rangle$$

Analysis of conditions (6.3) yields positive results with regard to the problem of the existence of periodic solutions of the problem for sufficiently small μ in the case of the commensurabilities 1)-3). For example, in the case of commensurability $n_l = 2n_g$ we have

$$\langle\Phi\rangle + \frac{\partial\langle K_2\rangle}{\partial\omega} = \Phi_{1,2}(\theta, \rho, \varphi)\sin(\omega + \lambda)$$
$$\Phi_{1.2}(\theta, \rho, \varphi) = D\sin 2\varphi\,[\sin^2\theta - \sin^2\rho\,(1 + \cos\theta + 2\cos^2\theta)]$$

and conditions (6.3) hold when $\omega + \lambda = 0, \pi$; $\Phi_{1.2}(0, \rho, \varphi) \neq 0$ (here $\cos\theta = a/K_0(a)$, $\cos\rho = H/K_0(a)$, H is an arbitrary constant and $D = \text{const} \neq 0$, φ, λ are the coordinates of the centre of mass of the body in the fixed coordinate system /3/). We have analogous formulas and arguments in the case of the commensurabilities 2), 3).

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Translated by L.K.

PMM U.S.S.R., Vol.51, No.2, pp.189-196, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

AVERAGING IN A QUASILINEAR SYSTEM WITH A STRONGLY VARYING FREQUENCY*

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The problem of the applicability of asymptotic averaging methods to single-frequency quasilinear systems are studied for the critical case. It is assumed that in the asymptotically large time interval under consideration the frequency (the derivative of the oscillation or rotational phase) is a slowly varying parameter allowing the singularity to be approximated by a power function of slow time or of a small parameter. The value of the frequence can vary strongly, can become arbitrarily small and equal to zero, and the "frequency" can even change its sign. Such situations arise when studying the oscillating and rotating systems, and particularly often in the problem of the control of specified objects /1/. The present

**Prikl.Matem.Mekhan.*, 51, 2, 244-252, 1987